J. Fluid Mech. (1965), *vol.* 23, *part* 4, *pp.* 767–778
Printed in Great Britain

Self-preserving development within turbulent boundary layers in strong adverse pressure gradients

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(Received 9 March 1965)

The development of a turbulent boundary layer in a strong adverse pressure gradient can be described by the two-layer model proposed by Stratford (1959), in which the outer part of the flow is assumed to be unmodified by the pressure-rise and the inner part described by two local parameters, the surface stress and the pressure gradient. The description suggests that the modification of the original flow is in some sense self-preserving, and it is shown here that self-preserving development of the modification is consistent with the Reynolds equations of turbulent flow in particular pressure distributions. For these distributions, the predictions of the two-layer model are confirmed without any need to make the sharp and arbitrary distinction between the two parts of the boundary layer.

1. Introduction

The concept of self-preserving development is important for the theory of turbulent flow, not only because it predicts properties of the rather special self-preserving flows but also because the self-preserving flows are asymptotic states which other, qualitatively similar, flows approach after long development. For example, boundary layers can develop in self-preserving fashion in particular adverse pressure gradients, and a layer which is not initially of the self-preserving form approaches that form with downstream development. It follows that the development in an arbitrary distribution of pressure could be calculated from the properties of the self-preserving layers so long as the layer structure always resembles closely the structure of the self-preserving layer appropriate to the current pressure gradient. This condition severely restricts the form of the pressure distribution, since the turbulent motion in the outer part of a boundary layer changes comparatively slowly in response to changes in pressure gradient. Rapid response is confined to the flow near the surface, an observation which led Stratford (1959) to propose a two-layer model for development in strong adverse pressure gradients. In the outer part, the turbulent motion is supposed unmodified by the pressure rise, but the inner part is assumed to be an equilibrium layer (Townsend 1961*a*) with motion determined by two parameters, the surface stress and the local pressure gradient. With the model, a good description of layer development is possible but neglect of the transition region between the completely adjusted flow in the true equilibrium layer and the unmodified flow at the outer edge of the boundary layer introduces some uncertainty in the validity of the predictions. A similar difficulty with the two-layer model used to describe the effects of a change of surface roughness has been resolved by showing that the two-layer model is a special case of a class of self-preserving flows and that the development is much the same whatever self-preserving flow model is used (Townsend 1965). The essential feature of these flows is that Reynolds stress and total head are very nearly constant along streamlines lying outside a critical surface whose approximate position can be set by considering the balance of turbulent energy. Along streamlines within the critical surface, total head and Reynolds stress undergo changes, and it is these changes that may be of self-preserving form in suitable conditions. The important difference between this class of self-preserving flow and the better-known self-preserving flows in jets and wakes is that the Reynolds stress does not become zero outside the flow.

The present problem is the development of a turbulent boundary-layer of considerable initial thickness within a region of adverse pressure gradient, especially in the early stages when the critical surface is much closer to the wall than the outer edge of the layer. The changes caused by the pressure field can be described completely by distribution functions describing the changes of total head and Reynolds stress along streamlines, and the functions must become very small outside the critical surface and take equilibrium forms very near the wall. Outside the critical surface, the modification of the flow is entirely an effect of increased separation of the streamlines consequent on the pressure rise and a displacement caused by the flow changes within the critical surface. It will be shown that, in certain pressure distributions, flows described by distribution functions of self-preserving form are dynamically consistent to the extent that they can satisfy the Reynolds equations for the mean velocity and the turbulent energy.

2. Nature of the flow

Consider a turbulent boundary layer on a flat surface, $z = 0$, with two-dimensional mean flow in the Ox-direction and subjected to an external pressure gradient. At $x = 0$, the boundary layer is assumed to have a general resemblance to one developing in zero pressure gradient with the same skin friction, so that the initial velocity distribution is of the logarithmic form

$$U_1 = u_1/k \log (z/z_1) \tag{2.1}$$

for values of z less than about one-sixth of the total thickness of the layer. Here u_1 is the friction velocity, k is the Kármán constant (approximately 0·41), z_1 is the roughness length of the surface. (N.B. The suffix 1 is used to denote conditions at $x = 0$, the suffix 0 for conditions at a positive value of x.) For simplicity, roughness lengths are used to describe the effect of the surface on the additive constant of the logarithmic distribution. For an aerodynamically smooth surface, $z_1 = \nu/u_1 \exp(-A) \approx 0{\cdot}1\nu/u_1$. For rough surfaces, z_1 is commonly about one-twentieth of the average height of the roughness elements. In both cases, the profile (2.1) is valid only for z/z_1 greater than a number of order one-hundred but the thickness of the excluded region is usually negligible. For example, in a boundary layer developing in zero pressure gradient with a skin friction coefficient of 3×10^{-3}, the ration of z_1 to the total thickness δ_0 is about 10^{-5} and the logarithmic profile is a good approximation to within $10^{-3}\,\delta_0$ of the wall.

Consistently with the assumption of a logarithmic profile and small height of the critical surface, the variation of the Reynolds stress can be neglected and the stress put equal to the surface value, u_1^2 in the kinematic units used in this paper. In fact, the fractional change is of order z/δ_0.

If the layer enters a region of strong pressure gradient, general arguments show that Reynolds stress is nearly unchanged along streamlines that lie above the critical surface $z = z_c(x)$ (Townsend 1961*b*). Since the stress is nearly independent of height at $x = 0$, the stress gradient remains negligible compared with the pressure gradient and total head is conserved along streamlines in the same region, i.e.

$$\left.\begin{aligned} \tfrac{1}{2}U^2 + P &= \text{fn}\,(\psi) \\ \tau &= u_1^2 \end{aligned}\right\} \quad \text{for} \quad z \gg z_c, \tag{2.2}$$

where P is the (kinematic) pressure rise from $x = 0$ and ψ is the stream function. We now make the self-preserving assumption that

$$\psi = \psi_1(P + \tfrac{1}{2}U^2) + \psi_s f(U/U_s), \tag{2.3}$$

where $$\psi_1(P + \tfrac{1}{2}U^2) = \frac{u_1 z_1}{k}\left[\frac{k}{u_1}(2P + U^2)^{\frac{1}{2}} - 1\right]\exp\left\{\frac{k}{u_1}(2P + U^2)^{\frac{1}{2}}\right\}$$

is the stream-function for $x = 0$ expressed as a function of total head. The second term describes the changes of total head caused by stress gradients and is negligible for large values of z/z_c or of U/U_s. The quantities, ψ_s and U_s, are scales of stream-function and velocity, depending only on x. The corresponding distribution of Reynolds stress is

$$\tau = u_1^2 + u_s^2 F(U/U_s), \tag{2.4}$$

where u_s^2 is a scale of Reynolds stress and the function $F(U/U_s)$, like $f(U/U_s)$, becomes small for large values of U/U_s. Figure 1 shows in diagrammatic form the flow system and an outline of the notation, and figure 2 the possible distributions of stream-function and Reynolds stress as functions of U.

The choice of the self-preserving forms has been made to satisfy the outer boundary conditions on the region of modified flow. Near the surface, the flow forms an equilibrium layer with a velocity distribution determined by the local stress-profile, and this requirement forms the inner boundary condition. For a linear variation of stress in the equilibrium layer,

$$\tau(x, z) = \tau(x, 0) + \alpha(x)z,$$

where α is the stress-gradient, and comparison with (2.4) shows that

$$u_1^2 + u_s^2 F(0) = u_0^2, \tag{2.5}$$

where u_0^2 is the local surface stress. In general, the velocity distribution satisfies conditions that are not simple, but they are sufficient, with the stress conditions, to determine completely the variations of the scales.

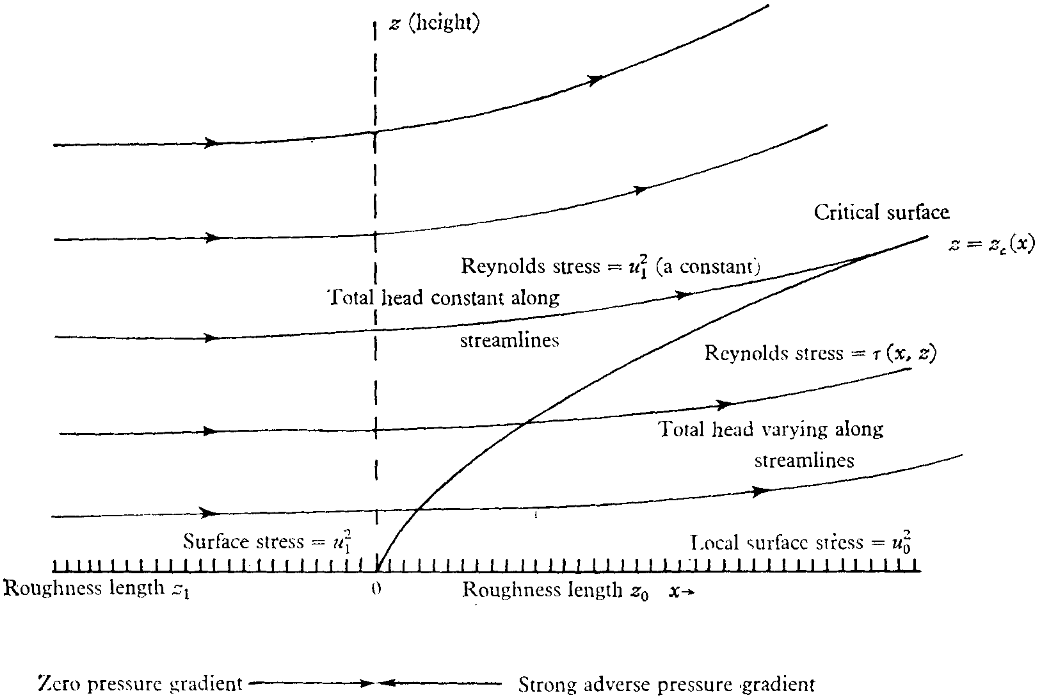

Figure 1. Flow diagram showing critical surface and indicating notation.

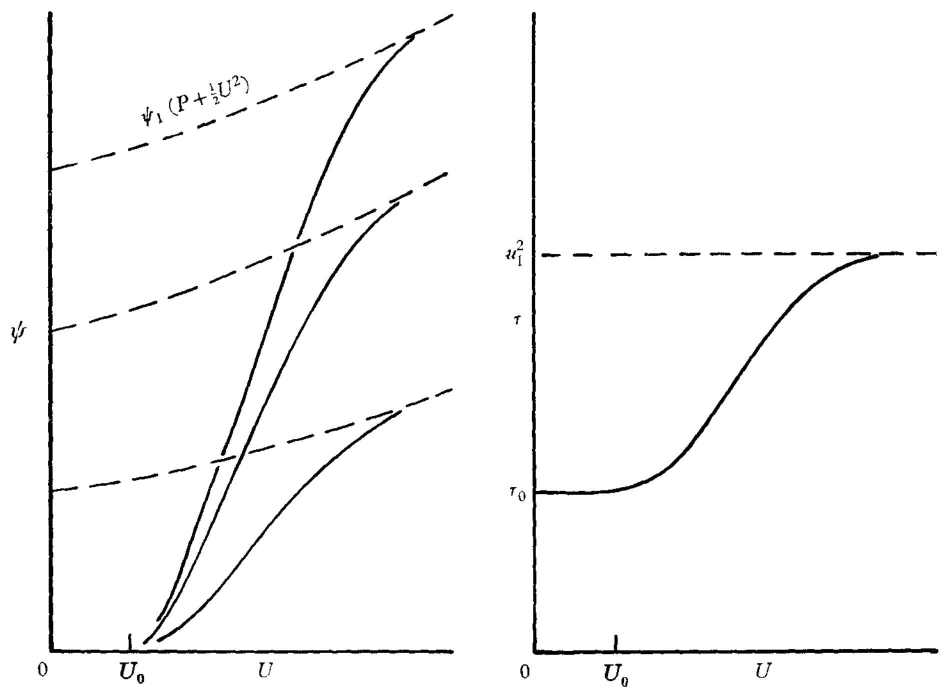

Figure 2. Self-preserving distributions of stream-function and Reynolds stress, expressed as functions of mean velocity. (N.B. Only the scale of stream-function varies during self-preserving development.)

3. Development with small surface-stress

For a linear variation of stress, there are theoretical and experimental grounds for believing that the velocity distribution in the equilibrium layer is given by

$$U = U_0 + \frac{2}{k_0}(\alpha z)^{\frac{1}{2}} \quad \text{for} \quad \alpha z \gg \tau_0, \tag{3.1}$$

where
$$U_0 = \frac{u_0}{k}\left[\log\left(\frac{4u_0^2}{\alpha z_0}\right) - 2(1-B)\right], \tag{3.2}$$

and
$$k_0 = k/(1-B) \approx 0{\cdot}50 \text{ (Townsend 1961}a\text{)}.$$

Supposing the greater part of the equilibrium layer to satisfy the condition of small surface stress, $\alpha z \gg u_0^2$, we have from (3.1) that

$$\begin{aligned}\psi &= U_0 z + \frac{4}{3k_0}\alpha^{\frac{1}{2}} z^{\frac{3}{2}} \\ &= \tfrac{1}{4}k_0^2\frac{U_0(U-U_0)^2}{\alpha} + \tfrac{1}{6}k_0^2\frac{(U-U_0)^3}{\alpha}.\end{aligned} \tag{3.3}$$

Choosing ψ_s so that $f(U_0/U_s) = -1$, comparison with the self-preserving form (2.3) shows that, necessarily,

$$\left.\begin{aligned}\psi_s &= \psi_1(P + \tfrac{1}{2}U_s^2), \\ \alpha\psi_s &= U_s^3, \\ U_0/U_s &= a,\end{aligned}\right\} \tag{3.4}$$

where a is a constant. Turning to the stress distribution,

$$\begin{aligned}\tau &= u_0^2 + \alpha z \\ &= u_0^2 + \tfrac{1}{4}k_0^2(U-U_0)^2,\end{aligned} \tag{3.5}$$

and for consistency with the self-preserving form,

$$\left.\begin{aligned}u_1^2 + u_s^2 F(U_0/U_s) &= u_0^2, \\ u_s/U_0 &= \text{const.}\end{aligned}\right\} \tag{3.6}$$

It is necessary to assume that the variation of $\psi_1(P + \frac{1}{2}U^2)$ is much less than that of ψ, which is nearly true if

$$\frac{1}{2}\frac{kU_s(U_0+U_s)}{u_1(2P+U_0^2)^{\frac{1}{2}}}\frac{U-U_0}{U_s} \ll 1 \tag{3.7}$$

for values of U in the equilibrium layer. The condition is satisfied if

$$(c_p/c_f)^{\frac{1}{2}} = P^{\frac{1}{2}}/u_1$$

is moderately large. From (3.4) and (3.6), the ratio u_s/U_s is constant and can be put equal to one.

The combined development conditions (3.4) and (3.6) require constancy of U_0 and u_0, although α is varying, which is consistent with equation (3.2) only if $u_0 = 0$, when $U_0 = 0$. If $k(2P + U_0^2)^{\frac{1}{2}}/u_1$ is moderately large, the rate of variation of U_0 for constant u_0 implied by the variation of α is comparatively small. To show this, use the development conditions (3.4) and (3.6) to write equation (3.2) as

$$U_0 = t(2P+U_0^2)^{\frac{1}{2}} + t\frac{u_1}{k}\left[\log\left\{\frac{4t^2(-F_a)^{\frac{3}{2}}z_1}{k(1-t^2)^{\frac{3}{2}}z_0}\left[\frac{k}{u_1}(2P+U_0^2)^{\frac{1}{2}}-1\right]\right\} - 2(1-B)\right], \quad (3.8)$$

where t is the stress-ratio parameter u_0/u_1, and $F_a = F(U_0/U_s)$. The second term varies much more slowly with P than the first and is much less than it for large values of $k(2P+U_0^2)^{\frac{1}{2}}/u_1$. For example, with $k(2P+U_0^2)^{\frac{1}{2}}/u_1$ about six, $t = 0{\cdot}3$, $-F_a = 1$, the ratio is one-tenth. Then, nearly

$$U_0^2 = [2t^2/(1-t^2)]P \quad \text{and} \quad P+\tfrac{1}{2}U_0^2 = P/(1-t^2). \quad (3.9)$$

It is shown in § 4 that $\alpha = dP/dx$ in self-preserving flow and so, from the relation $\alpha\psi_s = U_0^3$, it can be shown that

$$\frac{x}{P}\frac{dP}{dx} = \frac{2u_1}{k(2P+U_0^2)^{\frac{1}{2}}} \quad (3.10)$$

for large values of $k(2P+U_0^2)^{\frac{1}{2}}/u_1$. The large values of this parameter correspond with large Reynolds number of flow and, in these conditions, the streamwise variation of U_0 is slow although α varies rapidly. The conditions of constant U_0 and u_0 are then satisfied to an approximation similar to those used in the theory of self-preserving boundary layers.

To summarize, the development conditions set by the boundary conditions are

$$\left.\begin{aligned} U_0^2 &= [2t^2/(1-t^2)]P, \\ U_s^3 &= \alpha\psi_s = \alpha\frac{u_1 z_1}{k}\left[\frac{k}{u_1}(2P+U_0^2)^{\frac{1}{2}}-1\right]\exp\left\{\frac{k}{u_1}(2P+U_0^2)^{\frac{1}{2}}\right\}, \\ U_0/U_s &= a, \\ u_s^2 &= U_s^2 = u_1^2(1-t^2)/(-F_a). \end{aligned}\right\} \quad (3.11)$$

4. Dynamical consistency of the self-preserving flow

The Reynolds equation for the mean velocity is

$$U\left(\frac{\partial U}{\partial x}\right)_\psi + \frac{dP}{dx} = \frac{\partial\tau}{\partial z}.$$

From the self-preserving form for the velocity distribution (2.3), by differentiation with respect to x with constant ψ,

$$0 = \psi_1'\left[\frac{dP}{dx} + U\left(\frac{\partial U}{\partial x}\right)_\psi\right] + \frac{d\psi_s}{dx}f + \frac{\psi_s}{U_s}\left(\frac{\partial U}{\partial x}\right)_\psi f' - \frac{\psi_s}{U_s}\frac{dU_s}{dx}\frac{U}{U_s}f'. \quad (4.1)$$

Since U_s is constant during self-preserving development,

$$\left(\frac{\partial U}{\partial x}\right)_\psi = -\left(\psi_1'\frac{dP}{dx} + \frac{d\psi_s}{dx}f\right)\Big/\left(\psi_1' U + \frac{\psi_s}{U_s}f'\right). \quad (4.2)$$

At constant x,

$$\frac{\partial\tau}{\partial z} = \frac{\partial\tau}{\partial U}\frac{\partial U}{\partial\psi}\frac{\partial\psi}{\partial z} = \frac{U_s F' U}{U\psi_1' + \psi_s f'/U_s}, \quad (4.3)$$

and the equation for the mean velocity reduces to

$$\frac{\psi_s}{U_s}\frac{dP}{dx}f' - \frac{d\psi_s}{dx}Uf = U_s UF'. \tag{4.4}$$

To the approximation of small surface-stress, $f'(U_0/u_s) = 0$ from equation (3.3), and then

$$\alpha = \left(\frac{\partial \tau}{\partial z}\right)_{U=U_0} = U_s F'(U_0/U_s)\Big/\left(\frac{d\psi_s}{dx}\frac{dx}{dP}\right) = \frac{dP}{dx}. \tag{4.5}$$

Since $U_s^3 = \alpha\psi_s$, the final form of the equation of motion is

$$\eta F' = f' - \frac{1}{U_s}\frac{d\psi_s}{dx}\eta f \quad (\eta = U/U_s). \tag{4.6}$$

The coefficient $(1/U_s)\,(d\psi_s/dx)$ is of order $ku_1/P^{\frac{1}{2}}$ and, to the approximation in use, the last term is negligible. Then the Reynolds equation of mean flow is satisfied by self-preserving distributions of velocity and stress and the distribution functions are related by

$$\eta F' = f'. \tag{4.7}$$

Consistency with the Reynolds equation for the turbulent energy can be verified in a similar way. Using the analogous forms for the distributions of turbulent kinetic energy, lateral energy-flux and kinetic-energy dissipation rate,

$$\left.\begin{aligned} \tfrac{1}{2}\overline{q^2} &= \tfrac{1}{2}\overline{q_1^2} + U_s^2 F_T(U/U_s), \\ \frac{\partial}{\partial z}(\tfrac{1}{2}\overline{q^2 w} + \overline{pw}) &= U_s^4/\psi_s D(U/U_s), \\ \epsilon &= U_s^4/\psi_s E(U/U_s), \end{aligned}\right\} \tag{4.8}$$

the energy equation

$$U\left(\frac{\partial(\frac{1}{2}\overline{q^2})}{\partial x}\right)_\psi = \tau\frac{\partial U}{\partial z} - \frac{\partial}{\partial z}(\overline{pw} + \tfrac{1}{2}\overline{q^2 w}) - \epsilon$$

can be put into the form

$$U_s F_T' \frac{\alpha\psi_1' + (d\psi_s/dx)f}{\psi_1' + \psi_s/U_s^2 F} = \frac{u_1^2 + U_s^2 F}{\psi_1' + \psi_s/U_s^2 F} - \frac{U_s^4}{\psi_s}(E + D). \tag{4.9}$$

Now $d\psi_s/dx = \psi_s'\alpha$ and $\psi_1' \ll \psi_s/U_s^2$ in the current approximation, so nearly

$$\frac{\alpha\psi_s'}{U_s}(1+f)\frac{F_T'}{F} = \left(\frac{u_1^2}{U_s^2}\frac{1}{F} + 1\right) - E - D. \tag{4.10}$$

Since

$$\frac{\alpha\psi_s'}{U_s} = \frac{U_s^2\psi_s'}{\psi_s} = \frac{k^2 U_s^2}{u_1^2}\left(\frac{k}{u_1}(2P + U_0^2)^{\frac{1}{2}} - 1\right)^{-1}$$

and is small, the energy equation is also satisfied by self-preserving distributions. It is interesting that the term representing advection of kinetic energy is small everywhere.

5. Solutions of the development equations

The development equations are

$$\left.\begin{aligned} \alpha \frac{u_1 z_1}{k}\left\{\frac{k}{u_1}(2P+U_0^2)^{\frac{1}{2}}-1\right\}\exp\left[\frac{k}{u_1}(2P+U_0^2)^{\frac{1}{2}}\right] &= U_s^3, \\ U_0^2 &= \frac{2t^2}{1-t^2}P, \\ -U_s^2F(a) &= u_1^2(1-t^2), \\ a^2 = U_0^2/U_s^2 &= \frac{2t^2}{(1-t^2)^2}\frac{-F(a)P}{u_1^2}, \end{aligned}\right\} \tag{5.1}$$

where, by equation (4.7),

$$-F(a) = \int_a^\infty f'\frac{d\eta}{\eta}, \tag{5.2}$$

and depends on the velocity distribution function $f(\eta)$. All that is known about $f(\eta)$ is that it takes the form,

$$f(\eta) = \tfrac{1}{4}k_0^2 a(\eta-a)^2 + \tfrac{1}{6}k_0^2(\eta-a)^3 - 1$$

for small values of $\eta - a$, and approaches zero for large values of η. However, the dependence of $-F(a)$ on the form of $f(\eta)$ is weak. For example, for zero U_0 ($a = 0$), the most rapid approach to zero is obtained with the form

$$(a)\quad f(\eta) = \tfrac{1}{6}k_0^2\eta^3 - 1 \quad \text{for} \quad \eta \leqslant (6/k_0^2)^{\frac{1}{3}},$$
$$= 0 \quad \text{for} \quad \eta \geqslant (6/k_0^2)^{\frac{1}{3}}.$$

A very slow approach of total head to the undisturbed values, roughly as z^{-1}, is obtained with

$$(b)\quad f(\eta) = -\exp(-\tfrac{1}{6}k_0^2\eta^3).$$

For (a),

$$-F(0) = \tfrac{3}{2}(\tfrac{1}{6}k_0^2)^{\frac{1}{3}},$$

and for (b)

$$-F(0) = (\tfrac{2}{3})!\,\tfrac{3}{2}(\tfrac{1}{6}k_0^2)^{\frac{1}{3}} = 0{\cdot}903\,\tfrac{3}{2}(\tfrac{1}{6}k_0^2)^{\frac{1}{3}}.$$

The insensitivity to the form of the velocity distribution allows use of the simple form,

$$f(\eta) = \tfrac{1}{4}k_0^2 a(\eta-a)^2 + \tfrac{1}{6}k_0^2(\eta-a)^3 \quad \text{for} \quad \eta \leqslant \eta_0$$
$$= 0 \quad \text{for} \quad \eta \geqslant \eta_0,$$

where η_0 satisfies the equation

$$\tfrac{3}{2}a(\eta_0-a)^2 + (\eta_0-a)^3 = 6k_0^{-2}. \tag{5.3}$$

Then,

$$-F(a) = \tfrac{1}{4}k_0^2(\eta_0-a)^2 \tag{5.4}$$

and we proceed to calculate the development for a constant stress ratio. From the fourth of the development conditions (5.1),

$$\frac{2}{k_0}\frac{(-F_a)^{\frac{1}{2}}}{a} = \frac{u_1}{k_0}\left(\frac{2}{P}\right)^{\frac{1}{2}}\frac{1-t^2}{t}, \tag{5.5}$$

and $\frac{1}{2}k_0 a/(-F_a)^{\frac{1}{2}}$ may be calculated as a function of $-F(a)$ from equations (5.3) and (5.4) (table 1). For any value of t, $-F(a)$ can be found as a function of P

and the relation between pressure gradient and pressure rise obtained from

$$\frac{k_0^3 \alpha z_1}{8ku_1^2}\left(\frac{2(-F_a)^{\frac{1}{2}}}{k_0}\right)^3 (1-t^2)^{\frac{3}{2}}\left\{\frac{k}{u_1}\left(\frac{2P}{1-t^2}\right)^{\frac{1}{2}}-1\right\}\exp\left[\frac{k}{u_1}\left(\frac{2P}{1-t^2}\right)^{\frac{1}{2}}\right]=1. \quad (5.6)$$

$a = U_s/U_0$	$2(-F_a)^{-\frac{1}{2}}/k_0$	$ak_0/2(-F_a)^{\frac{1}{2}}$
0	2·884	0
0·2	2·788	0·0717
0·4	2·698	0·1483
0·6	2·614	0·2295
0·8	2·535	0·3156
1·0	2·462	0·4062
1·2	2·393	0·5015
1·4	2·328	0·6017
1·6	2·268	0·7054
1·8	2·211	0·8109
2·0	2·157	0·9272

N.B. It is assumed that $k_0 = 0{\cdot}50$.

TABLE 1

For $t = 0$, $8(-F_a)^{\frac{3}{2}}k_0^{-3} = 6/k_0^2$ and

$$\frac{3k_0}{4k}\frac{\alpha z_1}{u_1^2}\left\{\frac{k}{u_1}(2P)^{\frac{1}{2}}-1\right\}\exp\left[\frac{k}{u_1}(2P)^{\frac{1}{2}}\right]=1, \quad (5.7)$$

equivalent to a previous result for zero-stress development using a two-layer model (Townsend 1961*b*). Figure 3 shows the predicted relations between pressure gradient and pressure rise for values of t below 0·5. A curious feature is that the adverse pressure gradients are stronger for development with finite stress, $t \approx 0{\cdot}3$, than with zero surface-stress, $t = 0$. A similar behaviour is found with the two-layer model and receives some support from the observations of Schubauer & Klebanoff (see Townsend 1961*b*).

The conditions for the existence of the self-preserving development can now be put in more explicit form. They are (i) that $P^{\frac{1}{2}}/u_1$ should be large, (ii) that the surface stress is small enough to permit use of the velocity distribution (3.1), and (iii) that the thickness of the whole boundary layer is large enough to contain the modified flow. The second condition is that $\alpha z \gg \tau_0$ for most z within the modified layer. In terms of $(U-U_0)$, it is

$$\tfrac{1}{4}k_0^2(U-U_0)^2 \gg t^2u_1^2$$

and, substituting the value of $(U-U_0)$ at the edge, i.e. $(\eta_0-a)u_0$ and using equations (5.1) and (5.4), it becomes

$$1-t^2 \gg t^2 \quad \text{or} \quad t^2 \ll \tfrac{1}{2}. \quad (5.8)$$

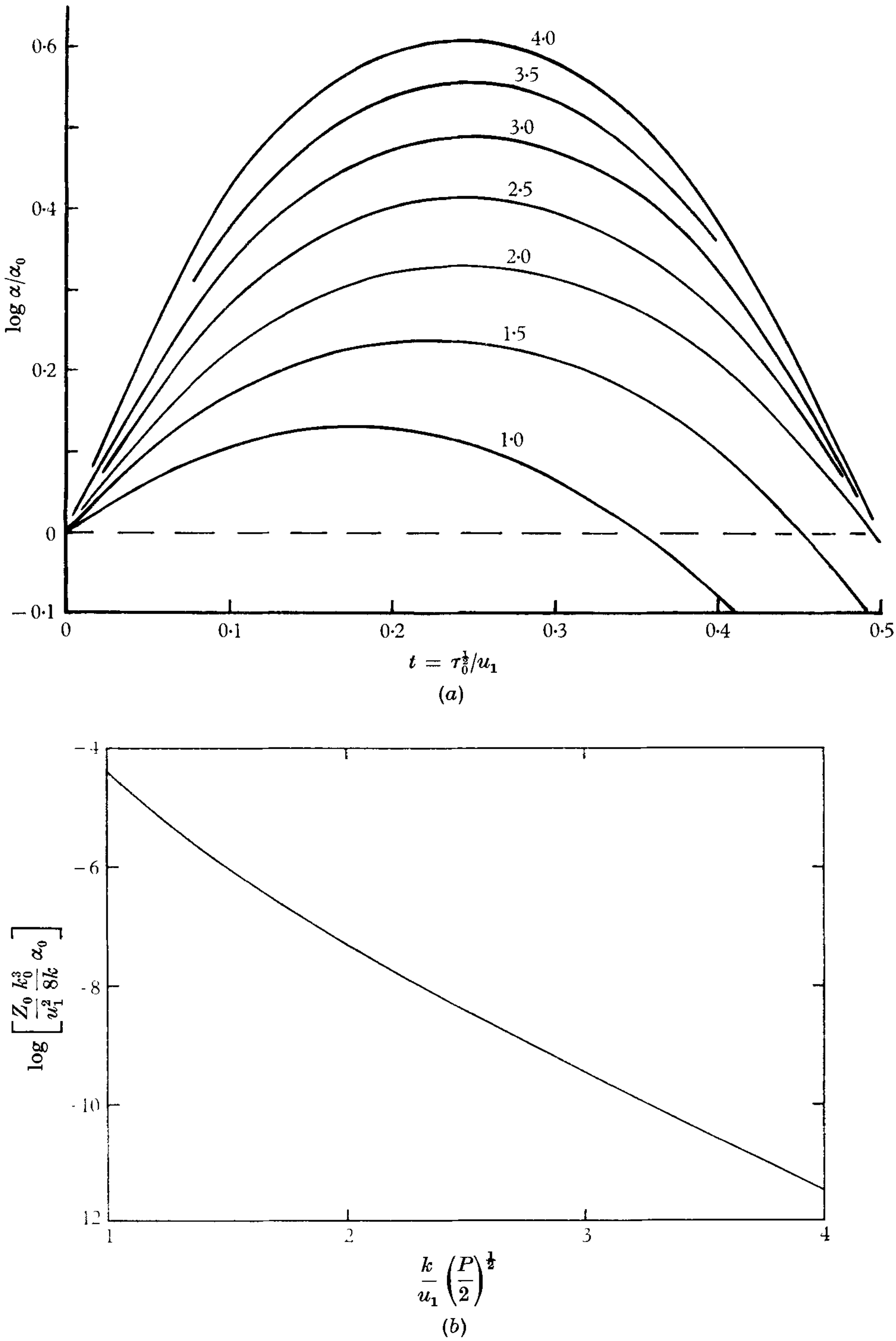

FIGURE 3. Variation of pressure gradient during self-preserving development. The numbers in (a) are the values of $(k/u_1)(\frac{1}{2}P)^{\frac{1}{2}}$ for the particular curve. α_0 is the pressure gradient for zero surface-stress. (N.B. $k_0 = 0{\cdot}50$).

The third condition is that the stream function at the edge of the modified layer should be considerably less than the stream function at the edge of the whole boundary layer, i.e.

$$\frac{u_1 z_1}{k}\left\{\frac{k}{u_1}(2P+U_e^2)^{\frac{1}{2}}-1\right\}\exp\left[\frac{k}{u_1}(2P+U_e^2)^{\frac{1}{2}}\right] \ll \frac{u_1 z_1}{k}\left\{\frac{kU_\infty}{u_1}-1\right\}\exp\frac{kU_\infty}{u_1}, \quad (5.9)$$

where $U_e = \eta_0 u_0 = (a(-F_a)^{-\frac{1}{2}}+2/k_0)(1-t^2)^{\frac{1}{2}}u_1$ is the velocity at the edge of the modified layer, and U_∞ is the free-stream velocity at $x = 0$. For small values of t, the condition is nearly that

$$\exp\left\{\frac{k}{u_1}\left[\left(2P+\frac{4u_1^2}{k_0^2}\right)^{\frac{1}{2}}-U_\infty\right]\right\} \ll 1,$$

or, let us say, less than e^{-q} where q is about two. Then the condition becomes

$$c_p + \frac{4k^2}{k_0^2}\gamma^2 \leqslant (1-\gamma q)^2, \quad (5.10)$$

where $c_p = 2P/U_\infty^2$ and $\gamma = u_1/(kU_\infty)$, and limits the coefficient of pressure rise to something less than one and usually about one-half. After this pressure rise, the modified region extends over most of the layer, and any self-preserving flow is an equilibrium flow of the kind described by Clauser (1956).

6. Discussion

The result of the analysis is that, in special pressure distributions, the boundary layer can develop in a way that is self-preserving in the sense used in § 2. The whole flow is very far from being self-preserving in the usual sense. Dynamical consistency of the self-preserving development implies that the development of a real flow in the same pressure gradient will be very similar, independently of the exact nature of the turbulent transfer process which determines the function specifying the velocity profiles. As in the similar change-of-roughness flows (Townsend 1965), the possible forms are restricted very severely by the imposed boundary-conditions, particularly by the necessity for a wall equilibrium layer of the mixing-length type. The two-layer model satisfies the conditions, and the predictions obtained by its use are hardly distinguishable from those obtained by any other plausible velocity distribution. Except for the academic interest in a new kind of self-preserving flow, the value of the work lies in the theoretical support for the usefulness of the two-layer approximation for pressure distributions of special kinds.

One of the family of pressure distributions allows development with continuously zero wall-stress, essentially similar to the flow discovered by Stratford (1959), and the distribution resembles closely that calculated by him using a two-layer model. The others refer to development with a small, constant wall-stress and have not been studied experimentally. All the pressure distributions are generally similar in form with a rapid decrease of pressure gradient (figure 3), and they have a qualitative resemblance to the pressure distributions found in boundary layers approaching separation. Remembering that the velocity and stress distributions in the self-preserving flows are the result of a settling-down process, it is plausible that the velocity and stress distributions in a separating layer may resemble closely the distributions in the self-preserving flow which has the same current values of pressure rise and surface stress. Then the two-layer model may be used to discuss separating flows with no more uncertainty about its accuracy than would be felt for the self-preserving flows.

Two small points should be mentioned. First, no self-preserving development is possible if the distribution of velocity in the equilibrium layer is logarithmic, as it will be if the surface stress is comparable with the initial stress. Secondly, the alternative self-preserving form

$$\psi = \psi_1(P + \tfrac{1}{2}U^2)\,[1 + f(U/U_s)]$$

may be used with essentially similar results. It must be used if the methods of this paper are to apply to the change of roughness flow, but no advantage is found over the more direct formulation (Townsend 1965).

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